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ASYMPTOTIC SOLUTIONS OF SOME PROBABILISTIC OPTIMAL CONTROL PROBLEMS

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An optimal control problem is analyzed for a stochastic dynamic system with the aim of maximizing the probability of hitting onto a fixed set at a finite instant. It is assumed that the set is a sphere of small radius. An irregular asymptotic expansion in powers of a small parameter — the sphere's radius — is constructed. Each term of this expansion is determined in an explicit analytic form. The approximate synthesis of the optimal control is found. Error estimates of the approximate method are proved. Examples are given. The problem on the probability of a controlled phase point hitting onto a small fixed neighborhood of a randomly moving point on the whole interval of motion has been studied earlier (see [1]). The present paper is akin to [2—5] with respect to the methods used.

1. Statement of the problem. Let a controlled motion be described by the system of equations

$$dx/dt = A(t)x + B(x, t)u + C(t)\xi(t), \quad x(t_0) = x_0 \qquad (1.1)$$

Here x is the n-dimensional phase coordinate vector, u is the m-dimensional control vector, $\xi(t)$ is the n-dimensional vector of random perturbations acting on the system, $A(t)$, $B(x, t)$ and $C(t)$ are matrices of dimensions $n \times n$, $n \times m$ and $n \times n$, respectively, with elements depending smoothly on t and x. It is assumed that matrix $C(t)$ is nonsingular for $t \in [t_0, T]$. The random perturbation vector is Gaussian white noise of unit intensity.

The constraints

$$u \in U \qquad (1.2)$$

are imposed on the controls, where U is a given bounded compact set in R^m. It is assumed that an exact measurement of the phase vector $x(t)$ of system (1.1) is possible at any instant $t \in [t_0, T]$. It is required to find a control method $u \in U$ maximizing the probability that the phase vector $x(t)$ hits onto the set

$$R_\varepsilon = \{x;\ (x_1^2 + \ldots x_n^2)^{1/2} \leqslant \varepsilon\} \qquad (1.3)$$

at a finite instant T. It is reckoned that the radius of sphere R_ε, dependent on a number ε, is fairly small.

Note. If the elements of matrix $A(t)$ depend smoothly on $t \in [t_0, T]$, system (1.1) can be reduced to the system

$$dx_1/dt = B(x_1, t)u + C(t)\xi(t), \quad x_1(t_0) = x_{1,0}$$

by a change of variables. Therefore, without loss of generality it can be assumed that $A(t) \equiv 0$ in (1.1).

Consider the Bellman function $S(x, t)$ of problem (1.1) — (1.3), equal to the maximum value of the probability of hitting onto set R_ε under the condition that the process starts from a phase vector $x(t) = x$ at an instant t. The Bellman equation for function S has the form [2 — 4]

$$S_\tau = \max_u H(x, \tau, S_x, u) + {}^1/_2 \operatorname{Sp}(C_1C_1'S_{xx}) \qquad (1.4)$$

$$\operatorname{Sp}(C_1C_1'S_{xx}) = \sum_{i,j=1}^{n} c_{ij}(\tau) S_{x_ix_j}$$

Here $T - t = \tau$ is reverse time, S_τ is the partial derivative with respect to τ, S_x is the vector of first partial derivatives, S_{xx} is the matrix of second partial derivatives of the function S with respect to the components of vector x, C_1 and B_1 are matrices obtained from matrices C and B by the substitution $t = T - \tau$.

The inequality

$$0 < \sum_{i,j=1}^{n} c_{ij}(\tau)\lambda_i\lambda_j \leqslant d_0|\lambda|^2, \quad |\lambda| \neq 0, \quad d_0 = \text{const} \qquad (1.5)$$

is valid since matrix C is nonsingular for all $\tau \in [0, T]$. The function S satisfies the boundary condition

$$S(x, 0) = \begin{cases} 1, & x \in R_\varepsilon \\ 0, & x \notin R_\varepsilon \end{cases} \qquad (1.6)$$

at the instant T of process termination, corresponding to the value $\tau = 0$.

As a result the problem (1.1)—(1.3) on determining the optimal control synthesis is reduced to solving a Cauchy problem for the nonlinear parabolic Eq. (1.4) with boundary condition (1.6) under the assumprion that a solution of Eq. (1.4) with (1.6) exists and is unique. An exact description of the class of such problems can be found in monograph [6]. The optimal control is determined after the determination of function S from the condition that the maximum is achieved in (1.4).

2. Small parameter method. Introduce the new variables

$$z_i = x_i/\varepsilon, \quad i = 1, 2, \ldots, n \qquad (2.1)$$

Equation (1.4) and condition (1.6) take the form

$$A(S; u) = -\varepsilon^2 S_\tau + \varepsilon \max_u H(z\varepsilon, \tau, S_z, u) + \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}(\tau) S_{z_iz_j} = 0 \qquad (2.2)$$

$$S(z, 0) = \psi(z) = \begin{cases} 1, & z \in R_1 \\ 0, & z \notin R_1, \quad R_1 = \{z; (z_1^2 + \ldots z_n^2)^{1/2} \leqslant 1\} \end{cases}$$

Note. The optimal control problem for a system in the presence of measurement errors for the phase vector $x(t)$ and the problem on maximizing the probability of hitting onto a fixed set at the instant $t = T$ if the intensity of the Gaussian white noise in (1.1) is a quantity of the order of ε^{-1} or if the possibility of control turns out to be small, both reduce this same mathematical problem [5].

Assume that a number $\alpha \geqslant 0$ exists such that the relation

$$H(z\varepsilon, \tau, S_z, u) = \varepsilon^\alpha H^\varepsilon(z, \tau, S_z, u)$$

is satisfied, where the function H^ε is bounded for all $\varepsilon > 0$ and for finite values of its arguments

$$H^\varepsilon(z, \tau, S_z, u) = \varepsilon^\alpha \sum^{n} u_i \sum_{j=1}^{n} b_{ij}^\varepsilon(z, \tau) S_{z_j}, \quad b_{ij}^\varepsilon = \varepsilon^{-\alpha} b(z\varepsilon, \tau)$$

Denote by $L^\varepsilon(S)$ the differential operator

$$L^\varepsilon(S) = -\varepsilon^2 S_\tau + \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}(\tau) S_{z_i z_j} \tag{2.3}$$

An approximate solution of problem (2.2) is sought as a sum of two functions

$$S^\circ(z,\tau;\varepsilon) + \varepsilon^{1+\alpha} S^1(z,\tau;\varepsilon) \tag{2.4}$$

The function S° is found by solving the boundary-value problem

$$L^\varepsilon(S^\circ) = 0, \quad S^\circ|_{\tau=0} = S|_{\tau=0} = \psi(z) \tag{2.5}$$

The function S^1 is determined as a solution of the problem

$$L^\varepsilon(S^1) + \max_u H^\varepsilon(z,\tau,S_z^\circ,u) = 0, \quad S^1|_{\tau=0} = 0 \tag{2.6}$$

It will be shown below that in certain cases it makes sense to seek the approximate solution of problem (2.2) in the form

$$S^\circ(z,\tau;\varepsilon) + \ldots\ \varepsilon^{j(1+\alpha)} S^j(z,\tau;\varepsilon) \tag{2.7}$$

where the functions S^k $(k = 2,3,\ldots,j)$ are determined recurrently from the solutions of the boundary-value problems

$$L^\varepsilon(S^k) + \max_u H^\varepsilon(z,\tau,S_z^{k-1},u) = 0, \quad S^k|_{\tau=0} = 0 \tag{2.8}$$

The functions $v^\circ, v^1, \ldots$ from U are determined by the relations

$$\max_u H^\varepsilon(z,\tau,S_z^j,u) = H^\varepsilon(z,\tau,S_z^j,v^j), \quad j=0,1,2,\ldots \tag{2.9}$$

The solution of each of the boundary-value problems (2.5) and (2.8) can be obtained in explicit analytic form. For this it suffices to write down the fundamental solution of Eq. (2.5)

$$p^\varepsilon(z-\lambda,\tau) = \varepsilon^n |C_0|^{-1}(2\pi)^{-n/2}\exp\left\{-\frac{\varepsilon^2}{2}\left[\sum_{i,j=1}^{n} c_0^{ij}(\tau)(z_i-\lambda_i)(z_j-\lambda_j)\right]\right\}$$

Here $c_0^{ij}(\tau)$ are the elements of the matrix inverse to the matrix $\|c_{ij}^\circ\|$, which is defined by the formula

$$c_{ij}^\circ = \int_0^\tau c_{ij}(\tau_1)\,d\tau_1$$

The function S° is found as a result of a convolution with respect to variable z

$$S^\circ(z,\tau;\varepsilon) = p^\varepsilon(z,\tau) * S^\circ(z,0;\varepsilon) = \int_{|\lambda|\leqslant 1} p^\varepsilon(z-\lambda,\tau)\,d\lambda \tag{2.10}$$

The functions S^k, $k = 1, 2, \ldots, j$ are determined by the formula

$$S^k(z,\tau;\varepsilon) = \int_0^\tau\int H^\varepsilon(\lambda,\tau_1,S_\lambda^{k-1}(\lambda,\tau_1;\varepsilon)v^{k-1})\,p^\varepsilon(z-\lambda,\tau-\tau_1)\,d\lambda\,d\tau_1 \tag{2.11}$$

$$d\lambda = d\lambda_1\ldots d\lambda_n$$

the integration with respect to λ in (2.11) is carried out over R^n. Certain properties of the functions S^k, $k = 0, 1, 2, \ldots, j$ should be noted.

Lemma. Let the coefficients b_{ij}^ε be of the linear form H^ε satisfy the inequality

$$|b_{ij}^\varepsilon(z,\tau)| \leqslant b_0(1 + b(\varepsilon)|z|^2), \quad b_0,\ b(\varepsilon) = \mathrm{const} \tag{2.12}$$

Then the bounds

$$|D^l S^\circ| \leqslant M_0 \varepsilon^{|l|} \tau^{-(n+|l|)/2} \exp\{-\varepsilon^2 \gamma_0 |z|^2/\tau\}, \quad |l| < 2 \quad (2.13)$$

$$D^l = \frac{\partial^{|l|}}{\partial z_i^{l_1} \partial z_j^{l_2}}, \quad l_1 = l_2 + |l|$$

$$|D^l S^k| \leqslant M_k \varepsilon^{k+|l|} \exp\{-\gamma_k \varepsilon^2 |z|^2/(\tau + \delta_k) + \varepsilon^2 \mu_k \tau\} \quad (2.14)$$

$$|l| \leqslant 1, \quad k = 1,2,\ldots,j$$

with certain constants M_k, γ_k, δ_k and μ_k are valid.

Proof. The derivatives with respect to variable z of the fundamental solution p^ε of boundary-value problem (2.5) satisfy [7] the bounds

$$|D^l p^\varepsilon(z-\lambda, \tau)| \leqslant \varepsilon^{|l|} \tau^{-|l|/2} m \varepsilon^n \tau^{-n/2} \exp\{-\gamma\varepsilon^2 |z-\lambda|^2/\tau\}, \quad |l| < 2$$

with certain constants m and γ depending on the coefficients $c_{ij}(\tau)$ in the last relation of (1.4). Therefore

$$|D^l S^\circ| \leqslant \varepsilon^{|l|} \tau^{-|l|/2} I(z, \tau, \varepsilon)$$

where

$$I(z, \tau, \varepsilon) = m\varepsilon^n \tau^{-n/2} \int_{|\lambda| \leqslant 1} \exp\left\{\frac{-\gamma\varepsilon^2 |z-\lambda|^2}{\tau}\right\} d\lambda$$

The equality

$$I(z, \tau, \varepsilon) = m\varepsilon^n \tau^{-n/2} \exp\left\{\frac{-\varepsilon^2\gamma|z|^2}{\tau}\right\} \times \int_{|\lambda| \leqslant 1} \exp\left\{-\varepsilon^2\gamma\left(|\lambda|^2 - 2\sum_{i=1}^n \frac{\lambda_i z_i}{\tau}\right)\right\} d\lambda$$

is valid. When $|\lambda| \leqslant 1$ the maximum value of the form

$$\sum_{i=1}^n \lambda_i z_i$$

equals

$$\sum_{i=1}^n |z_i|$$

hence when $|z| > 4$

$$I(z, \tau, \varepsilon) \leqslant M_0' \exp\left\{-\varepsilon^2\gamma\left(|z|^2 - 2\sum_{i=1}^n \frac{|z_i|}{\tau}\right)\right\} \leqslant M_0' \exp\left\{-\frac{\varepsilon^2\gamma|z|^2}{2\tau}\right\}$$

Here M_0' is a constant such that $I(0, \tau, \varepsilon) \leqslant M_0'$. On the other hand, when $|z| \leqslant 4$ and $\tau \in [0, T]$ the integral $I(z, \tau, \varepsilon)$, as a function of z, τ and ε, is bounded. Therefore, a constant $M_0 > M_0'$ can be chosen so as to satisfy inequality (2.13).

Let us prove inequality (2.14) with $k = 1$ for the function S^1. The estimate

$$|H^\varepsilon(z, \tau, S_z^\circ, v^\circ)| \leqslant K_1 \varepsilon \tau^{-1/2} \exp\{-\varepsilon^2 \gamma_0 |z|^2/\tau\}, \quad K_1 = \text{const}$$

is valid by virtue of inequalities (2.12) and (2.13). At first we show that function S^1 and its first derivatives in z are bounded on the set $|z| = 0$ for all τ. Using inequality (2.13), from (2.11) we have

$$|S^1_{z_l}| \leqslant \int_0^\tau\int \sum_{i=1}^m |v_i^\circ| \sum_{j=1}^n |b_{ij}^\varepsilon(\lambda, \tau_1)\| S^\circ_{\lambda_j}(\lambda, \tau; \varepsilon)\| \times$$
$$P^\varepsilon_{z_l}(z-\lambda, \tau-\tau_1)|\, d\lambda d\tau_1 \leqslant$$
$$\leqslant K_2\varepsilon \int_0^\tau\int \tau_1^{-1/2}(\tau-\tau_1)^{-1/2}\varepsilon^n(\tau-\tau_1)^{-n/2}\exp\left\{\frac{-\gamma\varepsilon^2|z-\lambda|}{(\tau-\tau_1)}\right\} d\lambda d\tau,$$
$$l = 1, 2, \ldots, n$$

Since

$$\varepsilon^n(\tau-\tau_1)^{-n/2}\int \exp\left\{\frac{-\gamma\varepsilon^2|z-\lambda|^2}{(\tau-\tau_1)}\right\} d\lambda \leqslant K_3 \quad \text{when } |z| = 0$$

the inequality

$$|S^1_{z_l}(0, \tau; \varepsilon)| \leqslant K_4\varepsilon \int_0^\tau \tau_1^{-1/2}(\tau-\tau_1)^{-1/2} d\tau_1, \quad l = 1, 2, \ldots, n$$

is valid with the constant $K_4 = K_3 K_2$. Set $\tau_2 = \tau_1 / \tau$. To within a constant the last integral takes the form

$$I_1 = \int_0^1 \tau_2^{-1/2}(1-\tau_2)^{-1/2} d\tau_2$$

This expression is the Euler integral of the first kind (beta-function) which can be expressed in terms of the Euler gamma-function

$$I_1 = \frac{[\Gamma(1/2)]^2}{\Gamma(1)} = \frac{\pi}{4}$$

As a result we obtain that the functions $S_{z_l}{}^1$ ($l = 1, \ldots, n$) are bounded for $|z| = 0$ and $\tau \in [0, T]$.

Let us set $S^1 = \varepsilon w \exp\{-\varepsilon^2\gamma_0|z|^2/[d_1(\tau+\delta_1)] + \varepsilon^2\mu_1\tau\}$, where d_1, δ_1 and μ_1 are constants to be chosen later. From (2.7) we have that the the relations

$$L^\varepsilon(w) + L^\varepsilon{}_1(w) + f_1(z, \tau, \varepsilon)\, w + \varepsilon^{-1}\exp\{+\varepsilon^2\gamma_0|z|^2 / [d_1(\tau+\delta_1)] - \varepsilon^2\mu_1\tau\}\, H^\varepsilon(z, \tau, S_z^\circ, v^\circ) = 0, \; w|_{\tau=0} = 0 \tag{2.15}$$

are valid for the function w. Here

$$L_1^\varepsilon(w) = -\frac{2\varepsilon^2\gamma_0}{d_1(\tau+\delta_1)}\sum_{i,j=1}^n c_{ij}(\tau)(z_j w_{z_i} + z_i w_{z_j})$$

$$f_1(z, \tau, \varepsilon) = \frac{4\varepsilon^4\gamma_0^2}{[d_1(\tau+\delta_1)]^2}\sum_{i,j=1}^n c_{ij}z_i z_j - \frac{\varepsilon^2\gamma_0|z|^2}{d_1(\tau+\delta_1)^2} - \frac{2\varepsilon^2\gamma_0}{d_1(\tau+\delta_1)}\sum_{i=1}^n c_{ii}(\tau) - \mu\varepsilon^2 \tag{2.16}$$

Since inequality (1.5) is satisfied, by choosing the constants $d_1 = 5d_0$ and $\mu_1 = 2c_0\gamma_0 / 5d_0\delta_1$, where c_0 is a constant such that

$$\left|\sum_{i=1}^n c_{ii}(\tau)\right| \leqslant c_0$$

we obtain that the function $f_1(z, \tau, \varepsilon) < 0$ for all z and τ. This enables us to apply to Eq. (2.15) the maximum principle [8] for parabolic equations on the set $|z| \geqslant \varepsilon_0 > 0$, $\tau \in [0, T]$. If the number ε_0 is fairly small, then by virtue of continuity on the set $|z| < \varepsilon_0$ the function w, as was shown above, is bounded: $|w| \leqslant K_5$, $K_5 \geqslant \pi K_4 / 4$. From the maximum principle it follows that function w is bounded on the set $|z| \geqslant \varepsilon_0$, $\tau \in [0, T]$ and inequality (2.14) is satisfied for $|l| = 0$. We now choose a constant $M_1 > K_5$ and a number δ_1 such that $\delta_1 \geqslant \varepsilon^2 \gamma_0 / 5d_0 (\ln M_1 - \ln K_5)$, then inequality (2.14) remains valid for all z and τ with $\gamma_1 = \gamma_0 / d_1$ for $|l| = 0$.

In order to obtain estimate (2.14) for the derivative of function S^1 it is necessary to differentiate Eq. (2.6) with respect to variable z and to repeat once more the arguments used above. The single differentiation with respect to z increases the order of estimate (2.14) in ε by unity. The estimates for the functions S^k, $k = 2, \ldots, j$ are obtained similarly, using estimate (2.14) with $|l| = 1$.

3. Error estimates for the approximate solution. Approximate synthesis of the optimal control. Denote $W^1 = S^\circ + \varepsilon^{(1+\alpha)} S^1$, where S° and S^1 are the functions obtained by formulas (2.10) and (2.11) as a result of solving the boundary-value problems (2.5) and (2.6). Here α is the positive number defined earlier in Sect. 2. Let us estimate the error yielded by the function W^1.

Theorem 1. Let condition (2.12) be satisfied. Then the estimate

$$|S - W^1| \leqslant K\tau\varepsilon^{4+2\alpha} \exp\{-\varepsilon^2\gamma_1 |z|^2 / (\tau + \delta_1) + \varepsilon^2 \mu_1 \tau\} \qquad (3.1)$$

is valid for function W^1. Here function S is a solution of the Bellman Eq. (2.2); γ_1, δ_1 and μ_1 are the constants occurring in (2.14) and K is a constant.

Proof. Let us set $S = W^1 + \omega$. Taking the notation in (2.2)—(2.4) into account, we obtain

$$0 = A(S; u) = A(S^\circ + \varepsilon^{1+\alpha} S^1 + \omega; u) = L^\varepsilon(S^\circ) + \varepsilon^{1+\alpha} L^\varepsilon(S^1) + L^\varepsilon(\omega) + \varepsilon^{1+\alpha} \max_u H^\varepsilon(z, \tau, S_z^\circ + \varepsilon^{1+\alpha} S_z^1 + \omega_z; u)$$

Since

$$\max_u H^\varepsilon(z, \tau, S_z^\circ + \varepsilon^{1+\alpha} S_z^1 + \omega_z; u) \leqslant H^\varepsilon(z, \tau, S_z^\circ, v^\circ) + \varepsilon^{1+\alpha} H^\varepsilon(z, \tau, S_z^1, v^1) + \max_u H^\varepsilon(z, \tau, \omega_z, u)$$

where v° and v^1 are the functions defined by relations (2.8), the inequality

$$0 = A(S; u) \leqslant L^\varepsilon(S^\circ) + \varepsilon^{1+\alpha} [L^\varepsilon(S^1) + H^\varepsilon(z, \tau, S_z^\circ, v^\circ)] + \varepsilon^{2(1+\alpha)} H^\varepsilon(z, \tau, S_z^1, v^1) + A(\omega; u) \qquad (3.2)$$

is valid By virtue of (2.5) and (2.6), from (3.2) follows

$$A(\omega; u) + \varepsilon^{2(1+\alpha)} H^\varepsilon(z, \tau, S_z^1 v^1) \geqslant 0 \qquad (3.3)$$

Using estimate (2.14) with $|l| = 1$, we obtain the validity of the inequality

$$|H^\varepsilon(z, \tau, S_z^1, v^1)| \leqslant K\varepsilon^2 g_1(z, \tau, \varepsilon) \qquad (3.4)$$
$$g_1(z, \tau, \varepsilon) = \exp\{-\varepsilon^2\gamma_1 |z|^2 / (\tau + \delta_1) + \varepsilon^2 \mu_1 \tau\}$$

We set $\omega = \omega_1 + K\tau\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon)$; then

$$A(\omega; u) + \varepsilon^{2(1+\alpha)} H^\varepsilon(z, \tau, S_z^1, v^1) \leqslant A(\omega_1; u^1) + f_2(z, \tau, \varepsilon) + K\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon) \qquad (3.5)$$

Here u^1 is a function from U such that

$\max_u H^\varepsilon(z, \tau, \omega_{1,z}, u) = H^\varepsilon(z, \tau, \omega_{1,z}, u^1)$

$f_2(z, \tau, \varepsilon) = f_1(z, \tau, \varepsilon)\, g_1(z,\tau,\varepsilon) + \max_u H^\varepsilon(z, \tau, g_{1,z}, u) - K\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon)$

Function f_1 has been defined by equality (2.16).

Similarly to what was done in the lemma when proving estimate (2.14), it can be shown that $f_2(z, \tau, \varepsilon) + K\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon) < 0$ for all z, τ and $\varepsilon > 0$. Therefore, the inequality

$$A(\omega_1; u^1) \geqslant 0, \quad \omega_1|_{\tau=0} = 0$$

follows from (3.4) and (3.5). Once again applying the maximum principle [8] to the parabolic operator $A(\omega_1; u^1)$, we have that $\omega_1 \leqslant 0$. Hence follows the inequality

$$S - W^1 \leqslant K\tau\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon) \tag{3.6}$$

On the other hand, let u^* be the optimal control of the original problem; then

$$0 = A(S; u^*) \geqslant A(S; v^\circ) \tag{3.7}$$

is valid. Here v° is the function obtained from (2.9) with $j = 0$. Using equalities (2.5) and (2.6), we obtain the validity of

$$A(W^1; v^\circ) - \varepsilon^{2(1+\alpha)} H^\varepsilon(z, \tau, S_z^1, v^\circ) = 0 \tag{3.8}$$

The inequality

$$A(W^1 - S;\ v^\circ) - \varepsilon^{2(1+\alpha)} H^\varepsilon(z, \tau, S_z^1, v^\circ) \geqslant 0 \tag{3.9}$$

follows from (3.6) and (3.8).

Consider the function $\omega_2 = W^1 - S - K\tau\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon)$, where K is the constant from (3.4). Then, the equality

$$A(W^1 - S;\ v^\circ) = A(\omega_2;\ v^\circ) + f_2(z, \tau, \varepsilon)$$

is satisfied. Just as before, it can be shown that $f_2(z, \tau, \varepsilon) + K\varepsilon^{4+2\alpha} g_1(z,\tau,\varepsilon) < 0$ for all z, τ and $\varepsilon > 0$; therefore, allowing for inequality (3.4), from (3.9) we obtain

$$A(\omega_2;\ v^\circ) \geqslant 0, \quad \omega_2|_{\tau=0} = 0$$

Applying the maximum principle again, we obtain

$$W^1 - S \leqslant K\tau\varepsilon^{4+2\alpha} g_1(z, \tau, \varepsilon) \tag{3.10}$$

Now (3.1) follows from (3.6) and (3.10).

Note. Condition (2.12) on the coefficients of the form H^ε is necessary for the application of the maximum principle [8].

Corollary 1. Let the inequality

$$|b_{ij}^\varepsilon(z, \tau)| \leqslant b_0\tau^{1/2}[1 + b(\varepsilon)|z|^2), \quad b_0,\ b(\varepsilon) = \text{const} \tag{3.11}$$

be satisfied instead of inequality (2.12). Then the estimate

$$|S - S^\circ| \leqslant K_0\,\tau\varepsilon^{2+\alpha} \exp\{-\varepsilon^2\gamma_0|z|^2/(\tau + \delta_0) + \varepsilon^2\mu_0\tau\} \tag{3.12}$$

is valid with constants γ_0, δ_0 and μ_0 and with some constant K_0.

The proof of estimate (3.12) is similar to the proof of estimate (3.1). Instead of inequality (3.4) it is necessary to use the inequality

$$| H^{\varepsilon} (z, \tau, S_z, {}^{\circ}v^1) | \leqslant K_0 \varepsilon \exp \{-\varepsilon^2 \gamma_0 | z |^2 / \tau\}$$

which follows from (2.13) with $| l | = 1$ and from inequality (3.11).

Note 4. It can be shown [9] that as $\varepsilon^n \tau^{-n/2} \to 0$ the function S° being a functional of the uncontrolled motion, decreases as a quantity proportional to $\varepsilon^n \tau^{-n/2}$. For small values of τ such that $\tau^{n/2} \varepsilon^{-n} \to 0$ the function S^c is a quantity of order of unity since the boundary condition $S^\circ (z, 0; \varepsilon) = \psi (z)$ is satisfied. Therefore, the estimate (2.13) is weaker as $\varepsilon^n \tau^{-n/2} \to 0$ and $n > 2$; however, it allows for the asymptotic behavior as $| z | \to \infty$, which is important for deriving estimate (3.1) under assumption (2.12). Asymptotics $\varepsilon^n \tau^{-n/2}$ would impose on the coefficients of form H^ε conditions of the kind of $| b_{ij}^{\varepsilon} (z,\tau) | \leqslant b (\varepsilon) \tau^{n/2}$ which are more restrictive than conditions (2.12) and (3.11).

Theorem 1 shows that the function W^1 well approximates the Bellman function S of the original problem. However, in certain cases the following approximations can be used.

Corollary 2. Let the hypothesis of Theorem 1 be valid and let the identities

$$v^\circ \equiv v^1 \equiv \ldots v^j, \qquad j = 1, 2, 3, \ldots \tag{3.13}$$

be fulfilled. Here the functions v^j, $j = 0, 1, \ldots$ are determined from relations (2.9). Then, the estimate

$$| S - W^j | \leqslant K_j \tau \varepsilon^{(j+1)(2+\alpha)} \exp \{-\varepsilon^2 \gamma_j | z |^2 / (\tau + \delta_j) + \varepsilon^2 \mu_j \tau\} \tag{3.14}$$

with constant K_j and constants γ_j, δ_j and μ_j from the estimate (2.14) is valid for the function $W^j = S^\circ + \varepsilon^{1+\alpha} S^1 + \ldots + \varepsilon^{j(1+\alpha)} S^j$ obtained when solving the boundary-value problems (2.5) — (2.8).

Proof. Consider the functions $\omega_j = S - W^j$. Similarly to inequality (3.2) we obtain

$$0 = A (S; u) \leqslant L^{\varepsilon} (S^\circ) + \sum_{i=1}^{j-1} \varepsilon^{i(1+\alpha)} [L^{\varepsilon} (S^j) + H^{\varepsilon} (z, \tau, S^i, v^i)] +$$
$$\varepsilon^{j(1+\alpha)} H^{\varepsilon} (z, \tau, S_z^j, v^j) + A (\omega_j; u)$$

By virtue of (2.6) — (2.9) we obtain the inequality

$$A (\omega_j; u) + \varepsilon^{j(1+\alpha)} H^{\varepsilon} (z, \tau, S_z^j, v^j) \geqslant 0$$

Hence, similarly to (3.5) we obtain the inequality

$$S - W^j \leqslant K_j \tau \varepsilon^{(j+1)(2+\alpha)} g_j (z, \tau, \varepsilon)$$

where K_j is a constant, whose existence is guaranteed by inequality (2.14)

$$| H^{\varepsilon} (z, \tau, S_z^j, v^j) | \leqslant K_j \varepsilon^{j+1} g_j (z, \tau, \varepsilon)$$

The function $g_j (z, \tau, \varepsilon)$ is determined similarly to the function g_1. The second inequality follows from (3.7) and the relation

$$A (W^j; v^\circ) - \varepsilon^{j(1+\alpha)} H^{\varepsilon} (z, \tau, S_z^j, v^\circ) = 0$$

valid when condition (3.11) and equalities (2.6) — (2.9) are satisfied.

The constructed asymptotic approximations W^1 and W^j, $j = 2, 3, \ldots$ do not answer the question on what the synthesis of the original problem's optimal control should be.

We show that the control v° found from relation (2.9) with $j = 0$, is nearly optimal in the sense of the proximity of the corresonding functionals. Let G denote a function which is a solution of the boundary-value problem

$$A\ (G;\ v^{\circ}) = 0, \quad G\,|_{\tau=0} = \psi\,(z) \tag{3.15}$$

Theorem 2. Let condition (2.12) be satisfied. Then the estimate

$$0 \leqslant S - G \leqslant 2K\tau\varepsilon^{4+2\alpha} \exp\{-\varepsilon^2\gamma_1\,|\,z\,|^2 / (\tau + \delta_1) + \varepsilon^2\mu_1\tau\}$$

is valid with the constants K, γ_1, δ_1 and μ_1 from (3.1).

Proof. From inequality (3.7) and equality (3.15) it follows that

$$S - G \geqslant 0 \tag{3.16}$$

On the other hand, using (3.8) and (3.15), we obtain

$$A\ (W^1 - G;\ v^{\circ}) - \varepsilon^{2(1+\alpha)}\,H^{\varepsilon}\,(z,\ \tau,\ S_z^{\ 1},\ v^{\circ}) = 0$$

Just as in the proof of inequality (3.10) we have that

$$W^1 - G \geqslant K\tau\varepsilon^{4+2\alpha}g_1\,(z,\ \tau,\ \varepsilon) \tag{3.17}$$

From (3.17) and (3.6) follows

$$0 \leqslant S - G = (S - W^1) + (W^1 + G) \leqslant 2K\tau\varepsilon^{4+2\alpha}g_1\,(z,\ \tau,\ \varepsilon)$$

Corollary 3. Let identities (3.13) be fulfilled and let inequality (2.12) be valid. Then the estimate

$$0 \leqslant S - G \leqslant 2K_j\tau\varepsilon^{(j+1)(2+\alpha)} \exp\{-\varepsilon^2\gamma_j\,|\,z\,|^2 / (\tau + \delta_j) + \varepsilon^2\mu_j\tau\}$$

is fulfilled.

Note. 5. The results obtained remain valid even when the set R_ε is a parallelepiped with sides that are multiples of the value of ε or is a strip of width ε. In the latter case the change of variables (2.1) needs to be carried out only for a part of the variables.

Example. Let the controlled motion of a material point be described by the equation

$$d^2y\,/\,dt^2 = u + \xi, \quad |\,u\,| \leqslant 1, \quad t \in [0,\ T], \quad y\,(0) = y_0, \quad y^{\cdot}\,(0) = y_0^{\cdot}$$

where ξ is Gaussian white noise of unit intensity. We seek the synthesis of the optimal control maximizing the probability of hitting onto the set $|\,y\,| \leqslant \varepsilon$ at the instant $t = T$ and the value itself of this probability. We set $y = (T - t)x^{\cdot} + x$; then

$$\frac{dx}{dt} = (T - t)(u + \xi), \quad |\,u\,| \leqslant 1, \quad t \in [0,\ T], \quad x\,(0) = x_0$$

Such a change does not alter the functional of the final state since $y\,(T) = x\,(T)$.

The Bellman equation and the boundary condition, allowing for substituting (2.1), take the form

$$\varepsilon^2 S_\tau = \varepsilon\tau\,|\,S_z\,| + \frac{1}{2}\tau^2 S_{zz}, \quad S\,(z,\,0) = \begin{cases} 1, & |\,z\,| \leqslant 1 \\ 0, & |\,z\,| > 1 \end{cases}$$

According to (2.5) and (2.6) we find the functions S° and S^1, as well as the control v°

$$S^{\circ}\,(z,\,\tau;\,\varepsilon) = \frac{\varepsilon}{\sqrt{2\pi\tau}} \int\limits_{|\lambda| \leqslant 1} \exp\left\{\frac{\varepsilon^2\,(z - \lambda)^2}{2\tau}\right\} d\lambda$$

$$v^\circ = \operatorname{sign} S_z^\circ = \operatorname{sign}\{\exp(-\varepsilon^2(z+1)^2/2\tau)[1-\exp(\varepsilon^2 z/\tau)]\} = \begin{cases} -1, & z>0 \\ +1, & z<0 \end{cases}$$

$$S^1(z,\tau,\varepsilon) = \frac{\varepsilon^2}{2\pi}\int_0^{\tau}\int_{-\infty}^{+\infty} \frac{\tau_1^{1/2}}{\sqrt{\tau-\tau_1}} \exp\left\{\frac{-\varepsilon^2(\lambda+1)^2}{2\tau_1}\right\}\left[1-\exp\left\{\frac{\varepsilon^2\lambda}{\tau_1}\right\}\right]\times$$

$$\exp\left\{\frac{-\varepsilon^2(z-\lambda)^2}{2}(\tau-\tau_1)\right\} d\lambda\, d\tau$$

For the case being considered we find the values of the constants used in the lemma and in Theorem 1

$$M_0 = M_1 = (2\pi)^{-1/2}\exp\{-3\varepsilon^2\}, \quad \gamma_0 = {}^3/_8$$

$$K_2 = K_3 = K_4 = 1, \quad K_5 = \sqrt{\pi}\,/\,2\Gamma({}^3/_2), \quad d_0 \doteq c_0 = 1$$

We choose the constant K such that $K > \max\{K_5; M_0\}$, then $\delta_1 \geqslant 3\varepsilon^2/40\,(\ln K - \ln K_5)$, $\mu = {}^3/_{20}\delta_1$, $\gamma_1 = \gamma_0/5 = {}^3/_{40}$. The validity of the estimate

$$|\, S - (S^\circ + \varepsilon S^1)\,| \leqslant K\tau\varepsilon^4 \exp\{-3\varepsilon^2|z|^2/40\,(\tau+\delta_1) + 3\varepsilon^2\tau/20\delta_1\}$$

stems from Theorem 1.

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